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Sparse inertially arbitrary patterns

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An n-by-n sign pattern $\mathcal{A}$ is a matrix with entries in $\{+,-,0\}$. An n-by-n nonzero pattern $\mathcal{A}$ is a matrix with entries in $\{*,0\}$ where $*$ represents a nonzero entry. A pattern $\mathcal{A}$ is inertially arbitrary if for every set of nonnegative integers n_1, n_2, n_3 with $n_1 + n_2 + n_3 = n$ there is a real matrix with pattern $\mathcal{A}$ having inertia (n_1, n_2, n_3). We explore how the inertia of a matrix relates to the signs of the coefficients of its characteristic polynomial and describe the inertias allowed by certain sets of polynomials. This information is useful for describing the inertia of a pattern and can help show a pattern is inertially arbitrary. Britz et al. [T. Britz, J.J. McDonald, D.D. Olesky, P. van den Driessche, Minimal spectrally arbitrary sign patterns, SIAM J. Matrix Anal. Appl. 26 (2004) 257–271] conjectured that irreducible spectrally arbitrary patterns must have at least $2n$ nonzero entries; we demonstrate that irreducible inertially arbitrary patterns can have less than $2n$ nonzero entries.

1. Introduction

A *sign pattern* is an n-by-n matrix $\mathcal{A} = [\mathcal{A}_{ij}]$ with entries in $\{+,-,0\}$. The set of all real matrices with sign pattern $\mathcal{A}$ is the qualitative class $Q(\mathcal{A}) = \{A = [a_{ij}] \in M_n(\mathbb{R}) | sign(a_{ij}) = \mathcal{A}_{ij}$ for all $i, j\}$. A *nonzero pattern* is an n-by-n matrix $\mathcal{A} = [\mathcal{A}_{ij}]$ with entries in $\{*, 0\}$ and $Q(\mathcal{A}) = \{A = [a_{ij}] \in M_n(\mathbb{R}) | a_{ij} \neq$

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0 if and only if $\mathcal{A}_{ij} = *$ for all $i, j\}$. We use the term *pattern* when statements hold for both sign and nonzero patterns.

The *inertia of a set* W of complex numbers is the ordered triple $i(W) = (n_1, n_2, n_3)$, where n_1 is the number of elements of W with positive real part, n_2 is the number of elements with negative real part, and n_3 is the number of elements with zero real part. For brevity, we say *inertia of* $p(x)$, denoted by $i(p)$, when referring to the inertia of the set of roots of a polynomial $p(x)$. A set of polynomials P *allows inertia* $\mathbf{i}$ if there is some polynomial $p(x) \in P$ such that $i(p) = \mathbf{i}$. The *inertia of a matrix* A, denoted by $i(A)$, is the inertia of the set of eigenvalues of A. The *inertia of a pattern* $\mathcal{A}$ is $i(\mathcal{A}) = \{i(A) | A \in Q(\mathcal{A})\}$. We say a pattern $\mathcal{A}$ *allows inertia* $\mathbf{i}$ if $\mathbf{i} \in i(\mathcal{A})$. A pattern $\mathcal{A}$ is *inertially arbitrary* if $\mathcal{A}$ allows each inertia (n_1, n_2, n_3), with $n_1 + n_2 + n_3 = n$. We say a pattern $\mathcal{A}$ *allows a polynomial* p if p is the characteristic polynomial of some matrix $A \in Q(\mathcal{A})$. Throughout this paper, if p is the characteristic polynomial of a matrix A, we denote p by p_A. A pattern $\mathcal{A}$ is *spectrally arbitrary* if $\mathcal{A}$ allows every monic real polynomial $p(x)$ of degree n. If a pattern is spectrally arbitrary then it is also inertially arbitrary. A pattern $\mathcal{A}$ is *reducible* if there is a permutation matrix P such that

$$P^T \mathcal{A} P = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ O & \mathcal{A}_3 \end{bmatrix},$$

where $\mathcal{A}_1$ and $\mathcal{A}_3$ are square matrices of order at least one. A pattern is *irreducible* if it is not reducible.

The concepts of spectrally and inertially arbitrary patterns were introduced in [6] and the Nilpotent–Jacobian method involving the Implicit Function Theorem was used to demonstrate that certain patterns were spectrally arbitrary. In [1] it was demonstrated that any irreducible spectrally arbitrary sign pattern must have at least $2n - 1$ nonzero entries and conjectured that no fewer than $2n$ nonzero entries are possible. In [4], this $2n$-conjecture was extended to include nonzero patterns and demonstrated to be true for patterns of order at most four. In [5], the conjecture was confirmed for patterns up to order five. Regarding inertially arbitrary patterns, reducible nonzero patterns were found in [8] which are inertially arbitrary and have less than $2n$ nonzero entries. As for irreducible patterns, it was demonstrated in [2] that at least $2n$ nonzero entries are needed in an irreducible inertially arbitrary pattern for each order $n \leqslant 4$. In Section 4 of this paper, we demonstrate for $n \geqslant 5$, there are *irreducible* inertially arbitrary nonzero patterns with $2n - 1$ nonzero entries and for $n \geqslant 6$ there are also *irreducible* inertially arbitrary *sign* patterns with $2n - 1$ nonzero entries.

In order to accomplish our goals we extend some known results from [2,9]. In Section 2, we explore how the inertia of a real monic polynomial relates to the signs of some of the coefficients of the polynomial. In Section 3, we consider certain sets of polynomials and provide some inertias that each set allows. Using these techniques along with the Implicit Function Theorem, it is shown that the family of patterns provided in Section 4 allows any inertia (n_1, n_2, n_3) with $n_1, n_2 > 0$. To show that this pattern allows the remaining inertias with $n_1 = 0$ or $n_2 = 0$ alternate arguments are provided. We end the paper by introducing another class of irreducible inertially arbitrary patterns of order n with less than $2n$ nonzero entries.

2. Inertia conditions on polynomials

In this section we extend work of Kim et al. [9, Lemma 20] (rewritten in terms of the inertia of a polynomial). In particular, we explore how the inertia of a real polynomial

$$p(x) = x^n + r_1 x^{n-1} + \cdots + r_{n-1} x + r_n \tag{1}$$

relates to the signs of some of the coefficients of $p(x)$. We use the fact that if p is a polynomial of degree n with roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the coefficient r_k of x^{n-k} can be described by an *elementary symmetric function* (see for example [7, p. 41])

$$S_k = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}. \tag{2}$$

In particular, $r_k = (-1)^k S_k$. Note that when considering S_k, if there is a summand $\lambda_1\lambda_2 \cdots \lambda_k$ with $\lambda \in \{\lambda_1, \ldots, \lambda_k\}$ for some λ with $\mathrm{Im}(\lambda) \neq 0$ but $\overline{\lambda} \notin \{\lambda_1, \ldots, \lambda_k\}$, then one can combine that summand with the corresponding summand containing $\overline{\lambda}$. We can repeat this operation until each summand is real. We refer to this process as *cancelling imaginaries*.

The next lemma extends a result of Kim et al. [9]. In particular, parts (a), (b), (d), (e), as well as (c) odd are from [9, Lemma 20]. The proof in [9] describes p as a product of linear and irreducible quadratic factors to analyze the sign of r_k. By using elementary symmetric functions, we say more about the sign of r_k (equivalently $(-1)^k S_k$) for other inertias:

Lemma 1. *Let p be a real polynomial of degree n as defined in (1).*

(a) *If $i(p) = (0, n, 0)$, then $r_k > 0$ for each $k = 1, \ldots, n$.*
(b) *If $i(p) = (n, 0, 0)$, then $S_k > 0$.*
(c) *If $i(p) = (0, 0, n)$, then $r_k = 0$ for each k odd, and $r_k \geqslant 0$ for each k even. Further, for k even, if $r_k = 0$ then $r_j = 0$ for $j \geqslant k$.*
(d) *If $i(p) = (1, n-2, 1)$, then $r_{n-1} < 0$.*
(e) *If $i(p) = (n-2, 1, 1)$, then $S_{n-1} < 0$.*
(f) *For $1 \leqslant t \leqslant n-1$, if $i(p) = (t, 0, n-t)$, then $S_k > 0$ for $1 \leqslant k \leqslant t$, and $S_k \geqslant 0$ for $k > t$.*
(g) *For $1 \leqslant t \leqslant n-1$, if $i(p) = (0, t, n-t)$, then $r_k > 0$ for $1 \leqslant k \leqslant t$, and $r_k \geqslant 0$ for $k > t$.*
(h) *If $i(p) = (n_1, n_2, n_3)$ and n_3 is odd, then $r_n = 0$ and $(-1)^{n_1} r_{n-1} \geqslant 0$.*

Proof. Parts (a), (b), (d), (e), as well as (c) for k odd, were proven in [9, Lemma 20]. Suppose $i(p) = (0, 0, n)$ and k is even. Using elementary symmetric functions, r_k is equal to the sum of an even number of roots multiplied together. Hence, after cancelling imaginaries, each summand in (2) will be nonnegative. Thus $r_k \geqslant 0$. Further, $r_k = 0$ if and only if p has less than k nonzero roots. It follows that if $r_k = 0$, then $r_j = 0$ for $j \geqslant k$.

For part (f), note that after cancelling imaginaries, each summand in S_k is nonnegative. For $k \leqslant t$ there is at least one positive summand. Part (g) follows from (f) by replacing each λ_{i_j} in (2) with $-\lambda_{i_j}$ and recalling that $r_k = (-1)^k S_k$.

For part (h), note that if n_3 is odd then one of the roots of p is 0 and so $r_n = (-1)^n S_n = 0$. In addition, either S_{n-1} equals 0 or has sign $(-1)^{n_2}$. So, either r_{n-1} equals 0 or has sign $(-1)^{(n-1)+n_2} = (-1)^{n_1}$. □

3. On the inertia of a set of polynomials

For a polynomial $p(x)$ we write $r_k(p)$ to mean the coefficient of x^{n-k} in the polynomial $p(x)$, and $S_k(p)$ to mean $(-1)^k r_k(p)$. The degree of p is denoted by $\deg(p)$. Let $\mathcal{S}$ be a pattern of order n. It was observed in [9, Theorem 1], that if

$$P_n = \{p(x) | \deg(p) = n \text{ and } r_2(p) > 0\}$$

is a subset of the set of characteristic polynomials allowed by $\mathcal{S}$, then $\mathcal{S}$ is inertially arbitrary. Further, in [2], it was observed that if

$$R_n = \{p(x) | \deg(p) = n \text{ and } r_{n-1}(p) \neq 0\}$$

is a subset of the set of characteristic polynomials allowed by $\mathcal{S}$, then, for n odd, $\mathcal{S}$ is inertially arbitrary and for n even, $\mathcal{S}$ allows each inertia except possibly $(0, 0, n)$. In the next result, we consider the following sets of polynomials and provide lists of inertias that these subsets of P_n allow. Let

$$G_n^1 = \{p(x) | \deg(p) = n \text{ and } S_k(p) > 0, 1 \leqslant k \leqslant n-2\},$$
$$G_n^2 = \{p(x) | \deg(p) = n \text{ and } r_k(p) > 0, 1 \leqslant k \leqslant n-2\}$$

and

$$H_n = \{p(x) | \deg(p) = n, S_1(p) = 0 \text{ and } S_k(p) > 0 \text{ for } k \text{ even with } 1 \leqslant k \leqslant n-2\}.$$

Lemma 2. *For $n \geqslant 3$, G_n^1 allows every inertia (n_1, n_2, n_3) with $n_1 > 0$, G_n^2 allows every inertia (n_1, n_2, n_3) with $n_2 > 0$, and H_n allows every inertia (n_1, n_2, n_3) with both $n_1, n_2 > 0$.*

Proof. Suppose $n \geqslant 3$ and n_1, n_2, n_3 are nonnegative integers with $n_1 + n_2 + n_3 = n$. Let $p(x)$ have roots

$\alpha \pm \beta i$ both with algebraic multiplicity $\lfloor n_1/2 \rfloor$,
$-\gamma \pm \beta i$ both with algebraic multiplicity $\lfloor n_2/2 \rfloor$,
$0 \pm \beta i$ both with algebraic multiplicity $\lfloor n_3/2 \rfloor$,
α with algebraic multiplicity $\lceil n_1/2 \rceil - \lfloor n_1/2 \rfloor$,
$-\gamma$ with algebraic multiplicity $\lceil n_2/2 \rceil - \lfloor n_2/2 \rfloor$,
and 0 with algebraic multiplicity $\lceil n_3/2 \rceil - \lfloor n_3/2 \rfloor$.

Then for $\alpha, \beta, \gamma > 0$, we have $i(p) = (n_1, n_2, n_3)$. We will show that for suitable choices $\alpha, \beta, \gamma > 0$, the resulting polynomial p with the above roots belongs to either G_n^1, G_n^2 or H_n, depending on the inertia being considered.

For k even and $1 \leqslant k \leqslant n-2$, S_k can be viewed as a polynomial in β of degree k, where the coefficient of β^k is positive. For k odd and $1 \leqslant k \leqslant n-2$, S_k can be viewed as a polynomial in β of degree $k-1$, where the coefficient of β^{k-1} is

$$[\alpha f_1(k) - \gamma f_2(k)],$$

for some nonnegative numbers $f_1(k)$ and $f_2(k)$. Note that for $i = 1, 2$, $f_i(k) = 0$ if and only if $n_i = 0$.

If $n_1 > 0$, then let $\gamma = 1$ and fix α large enough so that for each k odd, $1 \leqslant k \leqslant n-2$, the coefficient of β^{k-1} of S_k is positive. By taking β to be sufficiently large, we can ensure that $S_k > 0$ for $1 \leqslant k \leqslant n-2$. Thus, in this case, G_n^1 allows the inertia (n_1, n_2, n_3) with $n_1 > 0$.

If $n_2 > 0$, then let $\alpha = 1$ and fix γ large enough so that for each k odd, $1 \leqslant k \leqslant n-2$, the coefficient of β^{k-1} of S_k is negative. By taking β to be sufficiently large, we can ensure that $r_k = (-1)^k S_k > 0$ for $1 \leqslant k \leqslant n-2$. Thus, in this case, G_n^2 allows the inertia (n_1, n_2, n_3) with $n_2 > 0$.

If $n_1, n_2 > 0$, then $S_1 = n_1\alpha - n_2\gamma$. Fix α and γ so that $S_1 = 0$. By taking β to be sufficiently large, we can ensure that $S_k > 0$ for k even and $1 \leqslant k \leqslant n-2$. Thus, in this case, H_n allows the inertia (n_1, n_2, n_3) with $n_1, n_2 > 0$. $\square$

In the next section the following set of polynomials will be useful. Let

$$U_n = \{p(x) | \deg(p) = n,\ r_1(p) = 0 \text{ and } r_4(p) > 0\}.$$

Corollary 3. *For $n \geqslant 6$, let $\mathcal{A}$ be a sign (or nonzero) pattern of order n that allows each polynomial in U_n. Then $\mathcal{A}$ allows all inertias except possibly $(k, 0, n-k)$, $(0, k, n-k)$, for all $1 \leqslant k \leqslant n$.*

Proof. Suppose $\mathcal{A}$ allows each polynomial in U_n. Since $x^{n-4}(x^2+1)^2 \in U_n$, $\mathcal{A}$ allows inertia $(0, 0, n)$. For $n \geqslant 6$, $H_n \subseteq U_n$, hence U_n allows each inertia (n_1, n_2, n_3) with $n_1, n_2 > 0$ by Lemma 2. $\square$

4. Inertially arbitrary patterns with $2n-1$ nonzero entries

In this section we describe an irreducible pattern of order n with less than $2n$ nonzero entries. In particular, let

$$\mathcal{W}_n^* = \begin{bmatrix} * & * & 0 & 0 & 0 & \cdots & 0 \\ * & 0 & * & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & * & 0 & \ddots & \vdots \\ 0 & * & 0 & 0 & * & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & * & 0 & \cdots & \cdots & 0 & * \\ 0 & * & 0 & \cdots & \cdots & 0 & * \end{bmatrix} \text{ and}$$

$$\mathcal{W}_n = \begin{bmatrix} + & - & 0 & 0 & 0 & \cdots & 0 \\ + & 0 & - & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & - & 0 & \ddots & \vdots \\ 0 & + & 0 & 0 & - & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & + & 0 & \cdots & \cdots & 0 & - \\ 0 & + & 0 & \cdots & \cdots & 0 & - \end{bmatrix}.$$

We will show that $\mathcal{W}_n^*$ is inertially arbitrary for $n \geqslant 5$ and $\mathcal{W}_n$ is inertially arbitrary for $n \geqslant 6$. A pattern $\mathcal{A}$ of order n is *potentially nilpotent* if there is a matrix $A \in Q(\mathcal{A})$ such that $A^n = 0$ (in which case we know that $\mathcal{A}$ allows inertia $(0, 0, n)$). Kim et al. [9] introduced a class of sign (and hence nonzero) patterns of odd order which are inertially arbitrary but not potentially nilpotent. We will show that $\mathcal{W}_n$ (and hence $\mathcal{W}_n^*$) provides another such class of patterns for all $n \geqslant 6$.

It was demonstrated in [5, Theorem 4.1, Case 6] that $\mathcal{W}_5^*$ is not potentially nilpotent. Below we use the same argument to show that $\mathcal{W}_n^*$ is not potentially nilpotent for $n \geqslant 5$. It follows that $\mathcal{W}_n$ is not spectrally arbitrary.

Lemma 4. *For $n \geqslant 5$, $\mathcal{W}_n^*$ is not potentially nilpotent, and hence $\mathcal{W}_n$ is not potentially nilpotent.*

Proof. Suppose $n \geqslant 5$ and $A \in Q(\mathcal{W}_n^*)$. Then

$$r_4(p_A) = r_1(p_A)a_{23}a_{34}a_{42} - a_{23}a_{34}a_{45}a_{52}.$$

Thus if $r_1(p_A) = 0$, then $r_4(p_A)$ is nonzero. On the other hand, note that if A were nilpotent, then p_A would be x^n. Thus $\mathcal{W}_n^*$ is not potentially nilpotent. $\square$

Since $\mathcal{W}_n^*$ (resp. $\mathcal{W}_n$) is not potentially nilpotent for $n \geqslant 5$, it is also not spectrally arbitrary, and hence $\mathcal{W}_n^*$ (resp. $\mathcal{W}_n$) does not allow every monic polynomial of degree n. Although $\mathcal{W}_n$ is not spectrally arbitrary, the Implicit Function Theorem can still be used to provide information about polynomials allowed by $\mathcal{W}_n$ (see [2, Lemma 3.2] and [9, Lemma 5]). Using this technique we will show that U_n defined in the last section is a subset of the polynomials allowed by $\mathcal{W}_n$ (and hence $\mathcal{W}_n^*$).

Lemma 5. *For $n \geqslant 5$ and any $r_2, \ldots, r_n \in \mathbb{R}$, with $r_4 > 0$, there exists $A \in Q(\mathcal{W}_n)$ such that $p_A(x) = x^n + r_2x^{n-2} + \cdots + r_{n-1}x + r_n$.*

Proof. For $c > 0$, since $A \in Q(\mathcal{W}_n)$ if and only if $cA \in Q(\mathcal{W}_n)$, and since

$$p_{cA}(x) = x^n + cr_1x^{n-1} + c^2r_2x^{n-2} + \cdots + c^{n-1}r_{n-1}x + c^nr_n,$$

it suffices to show that the result holds for $r_1 = 0$ and $(r_1, r_2, \ldots, r_n)$ arbitrarily close to the origin with $r_4 > 0$. Let $A \in Q(\mathcal{W}_n)$ be a matrix of the form

$$A=\begin{bmatrix} a & -1 & 0 & 0 & 0 & \cdots & 0\\ b & 0 & -1 & 0 & 0 & \cdots & \vdots\\ 0 & 0 & 0 & -1 & 0 & \ddots & \vdots\\ 0 & d_1 & 0 & 0 & -1 & \ddots & \vdots\\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0\\ 0 & d_{n-4} & 0 & \cdots & \cdots & 0 & -1\\ 0 & d_{n-3} & 0 & \cdots & \cdots & 0 & -c \end{bmatrix} \tag{3}$$

where $a, b, c, d_1, \dots, d_{n-3}$ are all positive real numbers. With $d_0 = 0$,

$$\begin{aligned} p_A(x) &= x^n + (c-a)x^{n-1} + (b-ac)x^{n-2} + (bc-d_1)x^{n-3}\\ &\quad + \sum_{k=1}^{n-4}(-1)^k(d_k(c-a) + acd_{k-1} - d_{k+1})x^{n-3-k}\\ &\quad + (-1)^n a(d_{n-3} - cd_{n-4}). \end{aligned} \tag{4}$$

Fix $r_1 = 0$, and $r_4 > 0$. Let $c = a$ and $d_2 = r_4$. We seek positive numbers a, b, and $d_1, d_3, d_4, \dots, d_{n-3}$ such that

$$\begin{aligned} b - a^2 - r_2 &= 0,\\ ab - d_1 - r_3 &= 0,\\ a^2 d_1 - d_3 - r_5 &= 0,\\ (-1)^3(a^2 d_2 - d_4) - r_6 &= 0,\\ \vdots\ \vdots\ &\vdots\\ (-1)^{n-4}(a^2 d_{n-5} - d_{n-3}) - r_{n-1} &= 0,\\ (-1)^n a(d_{n-3} - ad_{n-4}) - r_n &= 0. \end{aligned}$$

If $r_i = 0$ for all $i \neq 4, 1 < i \leqslant n$, then a solution to the above system of equations is $b = a^2, d_1 = a^3, d_i = a^2 d_{i-2}$ for all i where $3 \leqslant i \leqslant n-3$, and $a = \sqrt[4]{r_4}$.

Let $f_1 = b - a^2, f_2 = ab - d_1, f_i = (-1)^{i-1}(a^2 d_{i-2} - d_i)$, for all i where $3 \leqslant i \leqslant n-3$, and $f_{n-2} = (-1)^n a(d_{n-3} - ad_{n-4})$. Set $\mathbf{a} = (a, a^2, a^3, a^5, a^6, \dots, a^{n-1}) \in \mathbb{R}^{n-2}$. Then using the Implicit Function Theorem, it is sufficient to show that the Jacobian $\frac{\partial(f_1,\dots,f_{n-2})}{\partial(a,b,d_1,d_3,d_4,\dots,d_{n-3})}$ is nonzero when $(a, b, d_1, d_3, d_4, \dots, d_{n-3}) = \mathbf{a}$ in order to complete the proof.

Let $J_{n-2} = \frac{\partial(f_1,\dots,f_{n-2})}{\partial(a,b,d_1,d_3,d_4,\dots,d_{n-3})}\big|_{\mathbf{a}}$. Then

$$J_{n-2} = \det\begin{bmatrix} -2a & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0\\ a^2 & a & -1 & 0 & & & & \vdots\\ 2a^4 & 0 & a^2 & -1 & 0 & & & \vdots\\ -2a^5 & 0 & 0 & 0 & 1 & 0 & & \vdots\\ 2a^6 & 0 & 0 & a^2 & 0 & -1 & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ (-1)^n 2a^{n-2} & 0 & 0 & & 0 & (-1)^n a^2 & 0 & (-1)^{n+1}\\ (-1)^{n+1}a^{n-1} & 0 & 0 & \cdots & \cdots & 0 & (-1)^{n+1}a^2 & (-1)^n a \end{bmatrix}$$

We note that J_{n-2} is not zero by showing that the above matrix is row equivalent to a lower triangular matrix with nonzero entries on the main diagonal. In particular, starting at the bottom row, multiply each row by $1/a$ and then add it to the row above until you get to the third row from the top. Multiply the third row by $1/a^2$ and add it to the second row. Multiply the second row by $-1/a$ and add it to the first row to obtain the lower triangular matrix with main diagonal $[-4a, a, a^2, a, -a, a, -a, \ldots, (-1)^{n+2}a]$.

Thus for any $r_2, \ldots, r_n$ with positive r_4 sufficiently close to 0, there exist positive values $a, b, c, d_1, \ldots, d_{n-3}$ such that $p_A(x) = x^n + r_1x^{n-1} + r_2x^{n-2} + \cdots + r_{n-1}x + r_n$ with $r_1 = 0$ and $r_4 > 0$. $\square$

Note that Corollary 3 and Lemma 5 imply that for $n \geqslant 6$, $\mathcal{W}_n$ allows all inertias except possibly $(k, 0, n-k)$, $(0, k, n-k)$, for all $1 \leqslant k \leqslant n$. In the next three results, we will show that $\mathcal{W}_n$ allows these inertias.

Given H is a nonempty subset of $\{1, \ldots, n\}$ and matrix A of order n, let $A(H)$ denote the submatrix of A obtained by removing rows and columns indexed by H. Also, let $\mathcal{W}_n^{(n)}$ denote the pattern obtained from $\mathcal{W}_n$ by replacing the (n, n) entry by zero.

Lemma 6. *$\mathcal{W}_n$ allows inertia $(n, 0, 0)$, for all $n \geqslant 5$.*

Proof. For $n \geqslant 5$, we first show there is a matrix in $Q(\mathcal{W}_n^{(n)})$ with inertia $(n, 0, 0)$ via an induction argument. The following matrix in $\mathcal{W}_5^{(5)}$ has inertia $(5, 0, 0)$:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 5 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Assume that $n \geqslant 6$. Through induction, construct any $A \in Q(\mathcal{W}_n^{(n)})$ such that $i(A(\{n\})) = (n-1, 0, 0)$. Using equation (4) with $c = 0$, we see that $\det(A) > 0$. Further $\det(A(\{n\})) > 0$. Thus $\det(A) \det(A(\{n\})) > 0$. By [9, Theorem 2(a)] there exists some positive diagonal matrix D such that $i(DA) = (n, 0, 0)$. Thus, there is a matrix in $\mathcal{W}_n^{(n)}$ with inertia $(n, 0, 0)$.

Now pick $A \in Q(\mathcal{W}_n^{(n)})$ such that $i(A) = (n, 0, 0)$. For $\epsilon > 0$, let $A_\epsilon \in Q(\mathcal{W}_n)$ be the matrix obtained from A setting the (n, n) position to $-\epsilon$. Since the eigenvalues of A_ϵ are continuous functions of ϵ, there exists some sufficiently small ϵ such that $i(A_\epsilon) = (n, 0, 0)$. Thus, $\mathcal{W}_n$ allows inertia $(n, 0, 0)$ for all $n \geqslant 5$. $\square$

Lemma 7. *$\mathcal{W}_n$ allows inertia $(0, n, 0)$, for all $n \geqslant 5$.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be a collection of any n negative real numbers. We will show that $\mathcal{W}_n$ allows the polynomial

$$p(x) = \prod_i (x - \lambda_i),$$

and hence $\mathcal{W}_n$ allows inertia $(0, n, 0)$. By Lemma 1, $r_1, \ldots, r_n$ are all positive. We first note that for the above polynomial $p(x)$,

$$r_1 r_{k-1} - r_k > 0, \tag{5}$$

for $3 \leqslant k \leqslant n-1$. This can be seen by using elementary symmetric functions:[3]

[3] To clarify the heart of the argument, we are suppressing notation: for example $\sum \lambda_i$ sums over all $i, 1 \leqslant i \leqslant n$; also $\sum \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_m}$ sums over $1 \leqslant i_1 < \cdots < i_m \leqslant n$; and $\sum \lambda_{i_1}^2 \lambda_{i_2}\lambda_{i_3}\cdots\lambda_{i_{k-1}}$ adds each summand such that $1 \leqslant i_2 < \cdots < i_{k-1} \leqslant n$ and $1 \leqslant i_1 \leqslant n$ with $i_1 \neq i_m$ for $2 \leqslant m \leqslant k-1$.

$$
\begin{aligned}
r_1 r_{k-1} - r_k > r_1 r_{k-1} - k r_k \\
&= (-1)^k \left(\sum \lambda_i \sum \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{k-1}} - k \sum \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \right) \\
&= (-1)^k \sum \lambda_{i_1}^2 \lambda_{i_2} \cdots \lambda_{i_{k-1}} \\
&> 0
\end{aligned}
$$

since each λ_i is a negative real number.

Given matrix A of the form (3) the characteristic polynomial is defined by (4). Let

$$
\begin{aligned}
c &= a + r_1, \\
b &= ac + r_2, \\
d_1 &= cb - r_3, \\
d_2 &= r_1 d_1 + r_4, \\
d_3 &= r_1 d_2 + acd_1 - r_5, \\
&\vdots \\
d_{n-3} &= r_1 d_{n-4} + acd_{n-5} + (-1)^{n-1} r_{n-1}
\end{aligned}
$$

be written as polynomials in a, where a is a root of the polynomial $f(a) = (-1)^n (ad_{n-3} - acd_{n-4}) - r_n$. We will show that f has a positive root and that such a value of a will ensure that $c, b, d_1, \ldots, d_{n-3}$ are all positive.

Note that as polynomials in a, d_1 has leading term a^3, and d_2 has leading term $r_1 a^3$. Further, it follows that for k even, d_k has leading term αa^{k+1} for some positive $\alpha \in \mathbb{R}$, and for k odd, d_k has leading term a^{k+2}. It follows that $f(a)$ is a monic polynomial of degree n and hence for sufficiently large a, $f(a) > 0$. But $f(0) = -r_n < 0$. Therefore f has a positive root.

Next, we show that $c, b, d_1, \ldots, d_{n-3}$ are all polynomials in a with positive coefficients. Note that c, b are polynomials in a with positive coefficients. We will show d_k is a polynomial in a with positive coefficients for all $1 \leqslant k \leqslant n-3$. For k even, by induction $d_k = r_1 d_{k-1} + a^2 d_{k-2} + r_1 a d_{k-2} + r_{k+2}$, as a polynomial in a, has constant term of the form $\beta + r_{k+2}$ for some $\beta > 0$. For $k > 2$ odd, by induction, $d_k = r_1 d_{k-1} + a^2 d_{k-2} + r_1 a d_{k-2} - r_{k+2}$ has constant term of the form $r_1(\beta + r_{k+1}) - r_{k+2}$ for some $\beta > 0$. By (5) it follows that d_k has only positive coefficients whether k is even or odd. Thus, there exist positive $a, b, c, d_1, \ldots, d_{n-3} \in \mathbb{R}$ for which $i(A) = (0, n, 0)$. □

Lemma 8. *If* $1 \leqslant k \leqslant n-1$ *and* $n \geqslant 5$, *then* $\mathcal{W}_n$ *allows inertias* $(k, 0, n-k)$, *and* $(0, k, n-k)$.

Proof. Suppose $1 \leqslant k \leqslant 4$. We may assume $A \in Q(\mathcal{W}_n)$ has form (3). Setting the values of (a, b, c, d_1, d_2) as in the tables below demonstrates that $\mathcal{W}_5$ obtains each inertia $(k, 0, 5-k)$ and $(0, k, 5-k)$ with $1 \leqslant k \leqslant 4$.

(a,b,c,d_1,d_2)	$i(A)$	(a,b,c,d_1,d_2)	$i(A)$
$(2,35,1,68,204)$	$(1,0,4)$	$(1,6,2,8,12)$	$(0,1,4)$
$(2,15,1,20,20)$	$(2,0,3)$	$(1,10,2,15,30)$	$(0,2,3)$
$(2,18,1,25,30)$	$(3,0,2)$	$(1,8,2,12,20)$	$(0,3,2)$
$(2,18,1,24,24)$	$(4,0,1)$	$(1,12,2,18,36)$	$(0,4,1)$

For $n > 5$, let $d_i = c^{i-2} d_2$ for all $i, 3 \leqslant i \leqslant n-3$. By (4), any polynomial allowed by $\mathcal{W}_n$ will be of the form $p(x) = x^n + r_1 x^{n-1} + \cdots + r_5 x^{n-5}$. Thus, using the corresponding values of (a, b, c, d_1, d_2) from the above tables, $\mathcal{W}_n$ allows inertias $(k, 0, n-k)$ and $(0, k, n-k)$, $1 \leqslant k \leqslant 4$.

Suppose $k \geqslant 5$. By Lemmas 6 and 7, $\mathcal{W}_k$ allows inertias $(k, 0, 0)$ and $(0, k, 0)$. Let $A \in Q(\mathcal{W}_k)$ be of the form (3) with inertia $(k, 0, 0)$ (respectively $(0, k, 0)$). Construct $\widehat{A} \in Q(\mathcal{W}_n)$ with $a, b, c, d_1, \ldots, d_{k-3}$ as in A and with $d_i = c^{i-k+3} d_{k-3}$ for $k-2 \leqslant i \leqslant n-3$. Then $p_{\widehat{A}} = x^{n-k} p_A$. Thus $\widehat{A}$ will have inertia $(k, 0, n-k)$ (resp. $(0, k, n-k)$). □

If two patterns $\mathcal{A}$ and $\mathcal{B}$ are related via transpose, negation, diagonal similarity or permutation similarity, then $\mathcal{A}$ is inertially arbitrary if and only if $\mathcal{B}$ is inertially arbitrary. Thus we say two sign patterns are *equivalent* if one can be obtained from the other via some combination of transpose, negation, diagonal similarity and permutation similarity. We say $\mathcal{H}$ is a subpattern of an n-by-n pattern $\mathcal{S}$ if $\mathcal{H} = \mathcal{S}$ or $\mathcal{H}$ is obtained from $\mathcal{S}$ by replacing one or more nonzero entries by a zero. An irreducible pattern which is inertially arbitrary is *minimal* if no proper irreducible subpattern is inertially arbitrary.

Theorem 9. *$\mathcal{W}_n^*$ is a minimal inertially arbitrary nonzero pattern for all $n \geqslant 5$ and $\mathcal{W}_n$ is a minimal inertially arbitrary sign pattern for all $n \geqslant 6$. Further, there is no inertially arbitrary sign pattern with nonzero pattern $\mathcal{W}_5^*$ and for $n \geqslant 6$, up to equivalence, the sign pattern $\mathcal{W}_n$ is the only inertially arbitrary sign pattern with nonzero pattern $\mathcal{W}_n^*$.*

Proof. By Corollary 3 and Lemmas 5–8, $\mathcal{W}_n$ (and hence $\mathcal{W}_n^*$) is inertially arbitrary for $n \geqslant 6$.

Suppose $n = 5$. By Lemmas 6–8, $\mathcal{W}_5^*$ allows inertias $(k, 0, 5-k)$ and $(0, k, 5-k)$ for each $1 \leqslant k \leqslant 5$. The following list of polynomials of degree 5 each have $r_1 = 0$ and $r_4 > 0$ with the listed inertia:

$p(x)$	$i(p)$
$x(x^2+1)^2$	$(0,0,5)$
$(x^2+x+5/4)(x-1)(x^2+1)$	$(1,2,2)$
$(x^2+2x+17)^2(x-4)$	$(1,4,0)$
$x(x+1)^2(x-1)^2$	$(2,2,1)$
$(x+2)^3(x-3)^2$	$(2,3,0)$

Thus, by Lemma 5, $\mathcal{W}_5^*$ allows each of the inertias in the above table. Note that for $n = 5$, Lemma 1(c) forces $r_4 < 0$ for inertias $(1,1,3)$, $(1,3,1)$ and $(3,1,1)$, so that Lemma 5 cannot be used in this case. But the matrices

$$\begin{bmatrix} -3 & -2 & 0 & 0 & 0 \\ -1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 & -3 & 0 & 0 & 0 \\ -3 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -3 \end{bmatrix}$$

illustrate that $\mathcal{W}_5^*$ allows inertias $(1,1,3)$ and $(1,3,1)$. Since $(n_1,n_2,n_3) \in i(\mathcal{W}_5^*)$ if and only if $(n_2,n_1,n_3) \in i(\mathcal{W}_5^*)$, the pattern $\mathcal{W}_5^*$ is inertially arbitrary.

We next show $\mathcal{W}_n^*$ is minimal. We need to consider subpatterns of $\mathcal{W}_n^*$. Observe that the nonzero entries above the main diagonal must be nonzero, otherwise the pattern is reducible. Suppose $A \in Q(U)$ where U is an inertially arbitrary subpattern of $\mathcal{W}_n^*$. By diagonal similarity we may assume A has the form (3) where some of $a, b, c, d_1, \ldots, d_{n-3}$ are possibly zero. We will consider the coefficients of the characteristic equation defined in (4).

If $d_{n-3} = 0$ then A is reducible, thus $d_{n-3} \neq 0$. Suppose $d_{n-4} = 0$. Then $r_n \neq 0$ and A can not have an inertia (n_1,n_2,n_3) where n_3 is odd. Thus $d_{n-4} \neq 0$.

Next observe that $a \neq 0$, for otherwise $r_n = 0$ and so A would necessarily be singular. Also, $c \neq 0$ otherwise $r_1 = -a \neq 0$, in which case A cannot have inertia $(0,0,n)$ by Lemma 1(c). Also note that if $i(A) = (0,0,n)$ then $c = a$ and so $r_2 = b - a^2$. Thus we need $b \neq 0$, otherwise $r_2 < 0$ which violates Lemma 1(c).

We know $d_1 \neq 0$, otherwise $r_3 = bc \neq 0$ and A cannot have inertia $(0,0,n)$ by Lemma 1(c).

Suppose $d_2 = 0$. Then if $i(A) = (0,0,n)$ we would have $r_4 = 0$ and hence also $r_j = 0$ for $j > 4$ and j even by Lemma 1(c). This would inductively imply that $d_k = 0$ for all even k, contradicting the fact that d_{n-3} and d_{n-4} are nonzero.

Suppose $d_k = 0$ for some k with $2 < k < n-4$. In fact, let k be the smallest such integer and suppose $i(A) = (0,0,n)$. In this case $c = a$. Also note that $r_j = 0$ for all j odd and $r_j \geqslant 0$ for all j even by Lemma 1(c). For k odd, $d_k = 0$ implies $r_{k+2} = a^2 d_{k-2}$ and hence $d_{k-2} = 0$, contradicting the minimality of k. Suppose k is even. Then $d_k = 0$ implies $r_{k+2} = -a^2 d_{k-2}$ in which case we must have $d_{k-2} < 0$. But then, we could inductively show that $d_j < 0$ for even $j \leqslant k-2$. Hence $r_4 = d_2 < 0$, contradicting the fact that $r_4 \geqslant 0$. Thus $d_k \neq 0$ for $2 < k < n-4$.

Therefore A has the form (3) where $a, b, c, d_1, \ldots, d_{n-3}$ are nonzero. Hence $\mathcal{W}_n^*$ has no proper irreducible subpattern which is inertially arbitrary. Therefore $\mathcal{W}_n^*$ is minimally inertially arbitrary for $n \geqslant 5$. It follows that $\mathcal{W}_n$ is a minimal inertially arbitrary pattern for $n \geqslant 6$.

To prove the final claims, suppose $A \in Q(\mathcal{A})$ where $\mathcal{A}$ is an inertially arbitrary sign pattern with nonzero pattern $\mathcal{W}_n^*$. Let p_A be as in (1). Considering r_1 with Lemma 1(a) and (b), we know that a_{11} and a_{nn} can not both have the same sign. Thus via negation we can assume $a_{11} > 0$ and $a_{nn} < 0$. By diagonal similarity, we can assume A is of the form (3) where $a, c > 0$ and $b, d_1, \ldots, d_{n-3}$ are nonzero. We will consider the coefficients of the characteristic equation defined in (4).

Suppose $n = 5$. Using Lemma 1(c), considering coefficient r_3, we know $d_1 > 0$ and considering r_5, we know $d_2 > 0$. But then $\mathcal{A}$ does not allow inertia $(1, 1, 3)$, $(1, 3, 1)$ or $(3, 1, 1)$: each of these inertias require $r_5 = 0$, in which case $r_4 < 0$, contradicting Lemma 1(g). Thus there is no sign pattern with nonzero pattern $\mathcal{W}_5^*$ which is inertially arbitrary.

Suppose $n \geqslant 6$. By [3, Lemma 5.1], we know that $b > 0$. Since $(0, 0, n) \in i(\mathcal{A})$, it follows from Lemma 1(c) that (in this case $c = a$ and hence) $d_k > 0$ for $1 \leqslant k \leqslant n-3$. Thus $A \in Q(\mathcal{W}_n)$. Therefore, up to equivalence, $\mathcal{W}_n$ is the only possible inertially arbitrary sign pattern with nonzero pattern $\mathcal{W}_n^*$. $\square$

We end by introducing another inertially arbitrary pattern of order $2n$ which has $2n-1$ nonzero entries. Let

$$\mathcal{M}_n = \begin{bmatrix} + & 0 & - & 0 & \cdots & \cdots & 0 \\ 0 & 0 & - & 0 & & & \vdots \\ + & 0 & 0 & - & \ddots & & \vdots \\ 0 & + & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & + & 0 & & & 0 & - \\ 0 & + & 0 & \cdots & \cdots & 0 & - \end{bmatrix}$$

and let $\mathcal{M}_n^*$ be the nonzero pattern of $\mathcal{M}_n$. One can observe that $\mathcal{M}_n^*$ is not equivalent to $\mathcal{W}_n^*$ by considering the digraph structure of these patterns (see for example [5]). In particular, the 2-cycle in the digraph of $\mathcal{W}_n^*$ is incident to a vertex with indegree $n-2$, but the 2-cycle in $\mathcal{M}_n^*$ is not incident to a vertex with degree $n-2$.

Theorem 10. *$\mathcal{M}_n^*$ is a minimal inertially arbitrary nonzero pattern for $n \geqslant 5$, and $\mathcal{M}_n$ is a minimal inertially arbitrary sign pattern for $n \geqslant 6$. Further, there is no inertially arbitrary sign pattern with nonzero pattern $\mathcal{M}_5^*$ and for $n \geqslant 6$, up to equivalence, the sign pattern $\mathcal{M}_n$ is the only inertially arbitrary sign pattern with nonzero pattern $\mathcal{M}_n^*$.*

Proof. If $A \in Q(\mathcal{W}_n)$ and $B \in Q(\mathcal{M}_n)$ then by positive diagonal similarity A is defined by (3) and B is equivalent to

$$\begin{bmatrix} a' & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & -1 & 0 & & & \vdots \\ b' & 0 & 0 & -1 & \ddots & & \vdots \\ 0 & d_1' & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & d_{n-4}' & 0 & & & 0 & -1 \\ 0 & d_{n-3}' & 0 & \cdots & \cdots & 0 & -c' \end{bmatrix}$$

for some positive $a', b', c', d'_1, \ldots, d'_{n-3}$.

Set $a' = a, b' = b, c' = c, d'_1 = d_1, \ldots, d'_{n-3} = d_{n-3}$. Then we claim that B has the same characteristic polynomial as A:

$$p_A = \det \begin{bmatrix} \lambda - a & 1 & 0 & 0 & \cdots & \cdots & 0 \\ -b & \lambda & 1 & 0 & & & \vdots \\ 0 & 0 & \lambda & 1 & \ddots & & \vdots \\ 0 & -d_1 & 0 & \lambda & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & -d_{n-4} & 0 & & \ddots & \lambda & 1 \\ 0 & -d_{n-3} & 0 & \cdots & \cdots & 0 & \lambda + c \end{bmatrix}$$

and

$$p_B = \det \begin{bmatrix} \lambda - a & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda & 1 & 0 & & & \vdots \\ -b & 0 & \lambda & 1 & \ddots & & \vdots \\ 0 & -d_1 & 0 & \lambda & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & -d_{n-4} & 0 & & \ddots & \lambda & 1 \\ 0 & -d_{n-3} & 0 & \cdots & \cdots & 0 & \lambda + c \end{bmatrix}.$$

Considering cofactor expansion along the top row, note that $p_A = (\lambda - a)D_1 + bD_2$ where $D_1 = \det[(\lambda I - A)(\{1\})]$ and $D_2 = \det[(\lambda I - A)(\{1, 2\})]$. But, $\det[(\lambda I - B)(\{1\})] = D_1$ and $\det[(\lambda I - B)(\{1, 3\})] = D_2$. It follows that $p_B = (\lambda - a)D_1 + bD_2 = p_A$.

Thus B has the same characteristic polynomial as A and hence $i(\mathcal{W}_n) \subseteq i(\mathcal{M}_n)$. Similarly, $i(\mathcal{M}_n) \subseteq i(\mathcal{W}_n)$. In fact, it follows that

$$i(\mathcal{M}_n) = i(\mathcal{W}_n) \quad \text{and} \quad i(\mathcal{M}_n^*) = i(\mathcal{W}_n^*). \tag{6}$$

Thus by Theorem 9, $\mathcal{M}_n$ is inertially arbitrary for $n \geqslant 6$ and $\mathcal{M}_n^*$ is inertially arbitrary for $n \geqslant 5$. Further, by Theorem 9 and (6) there is no inertially arbitrary sign pattern with nonzero pattern $\mathcal{M}_5^*$ and for $n \geqslant 6$, up to equivalence, the sign pattern $\mathcal{M}_n$ is the only inertially arbitrary sign pattern with nonzero pattern $\mathcal{M}_n^*$.

Using a similar argument to that in Theorem 9, one can show that $\mathcal{M}_n^*$ is minimal for $n \geqslant 5$, and hence $\mathcal{M}_n$ is minimal for $n \geqslant 6$. $\square$

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